Fundamental Theorem of Asset Pricing on Measurable Spaces under Uncertainty*

Markus Leippold[†] and Meriton Ibraimi[‡]

May 11, 2013

Abstract

It is common in the financial mathematics literature to start by fixing a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, on which the underlying price process is defined. We depart from this route in that we do not fix the prior $\mathbb{P}$. Under very general assumptions, we recover the Fundamental Theorem of Asset Pricing in discrete time under either a multiple-priors or a prior-free setting. We only require that $(\Omega, \mathcal{F})$ is a measurable space, while the multiple priors can be non-equivalent. Furthermore, the initial price of our market model does not need to be constant, but only measurable.

Keywords: Fundamental Theorem of Asset Pricing; uncertainty; multiple priors; $(\alpha, \mathcal{R})$-good deals.

JEL classification: C00, G12.

2010 Mathematics subject classification: Primary 60G42; Secondary 46N30, 46E30, 47N10.

*We thank Freddy Delbaen, Josef Teichmann, and the participants at the finance seminar at the ETH Zurich for useful comments. Financial support by the Swiss Finance Institute (SFI), Bank Vontobel, and the National Center of Competence in Research "Financial Valuation and Risk Management" is gratefully acknowledged.

[†]University of Zurich, Institute of Banking and Finance, Plattenstrasse 14, 8032 Zurich, Switzerland; markus.leippold@bf.uzh.ch; +41 (44) 634 50 69.

[‡]University of Zurich, Institute of Banking and Finance, Plattenstrasse 14, 8032 Zurich, Switzerland; meriton.ibraimi@bf.uzh.ch; +41 (44) 634 45 86.

1 Introduction

"If these things were so large, how come everyone missed them?" Queen Elizabeth asked this reasonable question after she had been given an academic briefing on the origins of the credit crunch at the renowned London School of Economics (LSE). The origins and effects of the crisis were explained to her by Professor Luis Garicano, director of research at the LSE's management department. When Garicano explained that "at every stage, [...] everyone thought they were doing the right thing", Her Majesty commented: "Awful."

It can be argued that one of the causes for the recent financial crisis was the overreliance on models that turned out to be inappropriate. However, we have to recall that any model, whether in finance or any other discipline, serves only as an approximation of the real world. As such, all models suffer from potential misspecification, often referred to as model uncertainty. This uncertainty has to be clearly distinguished from the concept of risk. Following Knight (1921), risk is present when future events occur with known probability, while uncertainty is present when the likelihood of future events is indefinite or incalculable. For many reasons, as commented by Roubini (2007), the recent financial crisis has to do with uncertainty rather than risk.[1]

The turmoil in financial markets has spurred academic interest for modeling uncertainty. Many of these endeavors in financial economics predominantly address uncertainty in terms of expected return ambiguity. Such ambiguity results in the formulation of a set of equivalent probability measures, i.e., these measures agree on the set of measure zero.[2] However, in the light of recent discussions on the role of uncertainty during the financial crisis, the presence of "black swan" events seems to call for a reformulation of basic financial theories under non-equivalent or singular measures.

While the financial economics literature on uncertainty is steadily growing, the literature in mathematical finance has remained surprisingly silent about the impact of uncertainty on arbitrage pricing theory. One of the cornerstones in mathematical finance is the Fundamental Theorem of Asset Pricing (FTAP). The FTAP was first formulated for a finite state space by Harrison and Pliska

[1] Many market observers would even argue that the increased uncertainty in financial markets is partly due to the increased opacity and lack of transparency, mainly as a result of "financial innovation." See, e.g., Roubini (2007).

[2] See, among many others, Hansen and Sargent (2001), Chen and Epstein (2002), Epstein and Schneider (2003), Leippold, Trojani, and Vanini (2008), Ju and Miao (2012), and Ulrich (2013). A notable exception and an extension to non-equivalent probability measures is the recent work by Epstein and Ji (2013).

(1981) and later generalized to measurable spaces and continuous-time processes by Delbaen and Schachermayer (1994). For a further account of the literature, we refer to Delbaen and Schachermayer (2006) and the references therein.

There is an important difference of the FTAP in the finite and infinite state space. In the finite case, we do not need to fix a prior probability measure to develop the theorem. In the infinite case, we first have to fix a prior probability measure as our real-world measure. The pricing measure is than taken to be equivalent to the fixed prior probability measure. Therefore, it is natural to ask if one can define arbitrage without fixing a prior probability measure and also obtain a version of the FTAP under a minimal set of assumptions.

The aim of this paper is to contribute to this research agenda by formulating a version of the discrete-time FTAP for a finite number of assets under a very general setup. In particular, we work on measurable spaces. Our version of the FTAP holds in the case of possibly non-equivalent multiple priors as well as in the no-prior case, i.e., under the absence of any prior assumption. We can also show that the no-prior assumption can be interpreted as a special case of the multiple-prior assumption. Furthermore, we remark that we only require our initial price to be measurable. Along our way, we introduce the concept of $(\alpha, \mathcal{R})$-good deals, which allows us the generalize the notion of arbitrage for our purpose.

Closely related to our contribution is the work by Riedel (2011) and Cherny (2007). These authors provide a definition of arbitrage without using any probability priors, but under more restrictive assumptions. To obtain a version of the FTAP in a one-period market, Riedel (2011) has to assume, e.g, that the underlying space is a Polish space and that the derivatives are continuous with respect to the metric. Using a similar definition of arbitrage as in Riedel (2011), Cherny (2007) derives a discrete-time and continuous-time version of the FTAP with a focus on its geometric characterization. In comparison, our setup is more general and allows us to derive the no-prior FTAP of Riedel (2011) and Cherny (2007) as special cases.

We remark that our analysis of the FTAP under uncertainty covers the different situations that may arise in financial models including uncertainty. For instance, we may consider a single investor who has uncertainty about which model to use, i.e., which prior to choose from a set of priors that may be non-equivalent. We may also cover a situation, in which multiple investors have non-equivalent beliefs. Finally, we may describe a situation, in which investors have no priors at all, i.e., when investors do not know at all what to believe. Indeed, we can think of this situation as a special case of a multiple-priors setting.

Our paper is structured as follows. In Section 2, we introduce arbitrage under the assumptions of multiple priors. In Section 3, we derive the FTAP without assuming the existence of any prior under the setup used in the previous section. We also show that the no-prior setting can be treated as a special case of the multiple-priors setting. Section 4 concludes.

2 FTAP under multiple priors

We first present the FTAP under multiple priors in a one-period market. We then extend the result to a multi-period setting.

2.1 One-period market

Let $(\Omega, \mathcal{F})$ be a measurable space and $\mathcal{F}_0 \subseteq \mathcal{F}$ a σ-subalgebra. Let $S_0 : \Omega \to \mathbb{R}^d$ be $\mathcal{F}_0$ measurable and $S_1 : \Omega \to \mathbb{R}^d$ be an $\mathcal{F}$-measurable map. From the financial point of view, we can regard S_t^i as the discounted price of the i-th asset at time $t = 0, 1$.[3] Denote by $\mathcal{M}_1(\Omega, \mathcal{F})$ the set of all probability measures on $(\Omega, \mathcal{F})$ and let $\mathcal{P} \subseteq \mathcal{M}_1(\Omega, \mathcal{F})$ be any subset. We call $\mathcal{P}$ the *set of priors*, i.e., the set of probability measures describing uncertainty about the real-world measure. We use the following definitions throughout the paper.

Definition 1: *A probability measure* $\mathbb{Q}$ *on* $(\Omega, \mathcal{F})$ *is called* a martingale measure *for the market model* $(\Omega, \mathcal{F}, S_0, S_1)$, *if it satisfies* $E_{\mathbb{Q}}[S_1 \mid \mathcal{F}_0] = S_0$, $\mathbb{Q}$-*a.s.*.

Definition 2: *We define the* support *of a Borel measure* μ *on* $\mathbb{R}^d$ *by*

$$\operatorname{supp}(\mu) = \bigcap_{A \subseteq \mathbb{R}^d \text{ closed, } \mu(A^c)=0} A. \tag{1}$$

[3] Without loss of generality, we do not explicitly model a riskless money market account.

Hence, supp(μ) *is the smallest closed set such that its complement has measure* 0*. For any probability measure* $\mathbb{P}$ *on* $(\Omega, \mathcal{F})$*, we denote by* $\mu_{\mathbb{P}}$ *the push-forward measure of* $\mathbb{P}$ *under the map* $S_1 - S_0 : \Omega \to \mathbb{R}^d$*.*

Definition 3: *The* relative interior *of a convex set* $C \subseteq \mathbb{R}^d$ *is the set of all points* $x \in C$*, such that for all* $y \in C$ *there exists some* $\varepsilon > 0$ *with*

$$x - \varepsilon(y - x) \in C. \tag{2}$$

The relative interior of C *is denoted* ri(C)*.*

In the following definition, we generalize the notion of an arbitrage opportunity for our purpose. Such a generalization is necessary, since in our framework we do not fix ex-ante a probability measure but allow for a set of non-equivalent measures. In addition, we replace the notion of arbitrage by the notion of a 'good deal' with a certain probability, say α.[4]

Definition 4 ($(\alpha, \mathcal{R})$-good deals): *Let* $\mathcal{R} \subseteq \mathcal{P}$ *and* $\alpha \in (0, 1]$*. We say there is an* $(\alpha, \mathcal{R})$*-good deal in the market model* $(\Omega, \mathcal{F}, S_0, S_1)$*, if there exists* $\pi \in \mathbb{R}^d$ *with*

$$\forall \mathbb{P} \in \mathcal{R} : \mathbb{P}(\pi \cdot S_1 \geq \pi \cdot S_0) \geq \alpha, \tag{3}$$

$$\exists \mathbb{P} \in \mathcal{R} : \mathbb{P}(\pi \cdot S_1 > \pi \cdot S_0) > 0. \tag{4}$$

We call the $(1, \mathcal{R})$*-good deal an* $\mathcal{R}$*-arbitrage.*

Under the above setup and with the definition of an $(\alpha, \mathcal{R})$-good deal, we can now state the FTAP under possibly non-equivalent multiple priors as follows.

Theorem 1 (One-period FTAP under multiple priors): *The following are equivalent:*

1. *The market* $(\Omega, \mathcal{F}, S_0, S_1)$ *is free of* $(\alpha, \mathcal{R})$*-good deals.*
2. *For any convex set* $C \subseteq \mathbb{R}^d$ *with* $\mathbb{P}(S_1 - S_0 \in \bar{C}) \geq \alpha$ *for all* $\mathbb{P} \in \mathcal{R}$*, we have* $0 \in \mathrm{ri}C$*.*
3. *For any convex set* $C \subseteq \mathrm{conv}(\bigcup_{\mathbb{P} \in \mathcal{R}} \mathrm{supp}\mu_{\mathbb{P}})$ *with* $\mathbb{P}(S_1 - S_0 \in \bar{C}) \geq \alpha$ *for all* $\mathbb{P} \in \mathcal{R}$*, there exists a martingale measure* $\mathbb{Q}$ *such that* $\mathrm{supp}\mu_{\mathbb{Q}} \subseteq \bar{C}$ *and* $\mathbb{Q}(S_1 - S_0 \in \mathrm{ri}C) > 0$*.*

[4] We thank Josef Teichmann sharing with us his definition of α-good deals.

Proof. We prove Theorem 1 as follows:

Step 1 ((1)⇒ (2)): We prove the contraposition. Let $C \subseteq \mathbb{R}^d$ be convex with $\mathbb{P}(S_1 - S_0 \in \bar{C}) \geq \alpha$ for all $\mathbb{P} \in \mathcal{R}$ and assume $0 \in \mathbb{R}^d \setminus \mathrm{ri}C$. For each $\mathbb{P} \in \mathcal{R}$, we have $\mu_{\mathbb{P}}(\bar{C} \cap \mathrm{supp}\mu_{\mathbb{P}}) \geq \alpha$.[5] Let $C' = \mathrm{conv}(\bigcup_{\mathbb{P}\in\mathcal{R}} \bar{C} \cap \mathrm{supp}\mu_{\mathbb{P}}) \subseteq C$, then we have $\mu_{\mathbb{P}}(\overline{C'}) \geq \alpha$ for all $\mathbb{P} \in \mathcal{R}$ and $0 \notin \mathrm{ri}C'$. Hence, by the separation theorem, there exists $\pi \in \mathbb{R}^d$ such that $\pi \cdot y \geq 0$ for all $y \in \mathrm{ri}C'$ and $\pi \cdot y^* > 0$ for some $y^* \in \mathrm{ri}C'$. Therefore, since y^* is a convex combination of elements in $\bigcup_{\mathbb{P}\in\mathcal{R}} \bar{C} \cap \mathrm{supp}(\mu_{\mathbb{P}})$, there must exist $y^{**} \in \bigcup_{\mathbb{P}\in\mathcal{R}} \bar{C} \cap \mathrm{supp}(\mu_{\mathbb{P}})$ with $\pi \cdot y^{**} > 0$. Consequently,

$$\forall \mathbb{P} \in \mathcal{R} : \mu_{\mathbb{P}}(\{y \in \mathbb{R}^d \mid \pi \cdot y \geq 0\}) \geq \alpha \text{ and } \exists \mathbb{P} \in \mathcal{R} : \mu_{\mathbb{P}}(\{y \mid \pi \cdot y > 0\}) > 0. \tag{5}$$

Hence, π is an $(\alpha, \mathcal{R})$-good deal and we have shown that (1) implies (2).

Step 2 ((2)⇒(1)): Assume (2). To achieve a contradiction, assume that there exists $\pi \in \mathbb{R}^d$ such that

$$\forall \mathbb{P} \in \mathcal{R} : \mu_{\mathbb{P}}(\{y \mid \pi \cdot y \geq 0\}) \geq \alpha \text{ and } \mu_{\mathbb{P}^*}(\{y \mid \pi \cdot y > 0\}) > 0 \text{ for some } \mathbb{P}^* \in \mathcal{R}. \tag{6}$$

Let $H = \{y \mid \pi \cdot y \geq 0\}$ and $C = \mathrm{conv}(\bigcup_{\mathbb{P}\in\mathcal{R}} H \cap \mathrm{supp}(\mu_{\mathbb{P}})) \subseteq H$. Since $\mu_{\mathbb{P}}(\bar{C}) \geq \alpha$ for all $\mathbb{P} \in \mathcal{R}$, we have by assumption (2), $0 \in \mathrm{ri}C$. By (6), there exists $y^* \in C$ with $\pi \cdot y^* > 0$. Since $0 \in \mathrm{ri}C$, there exists $\varepsilon > 0$ such that $-\varepsilon y^* \in C$. Therefore,

$$-\varepsilon y^* = \alpha_1 y_1 + \cdots + \alpha_n y_n, \tag{7}$$

for some $y_1, \ldots, y_n \in \bigcup_{\mathbb{P}\in\mathcal{R}} H \cap \mathrm{supp}\mu_{\mathbb{P}}$. It follows that

$$0 > -\varepsilon \pi \cdot y^* = \alpha_1 \pi \cdot y_1 + \cdots + \alpha_n \pi \cdot y_n, \tag{8}$$

contradicting the assumption that $\pi \cdot y \geq 0$ for all $y \in C \subseteq H$.

Step 3 ((3)⇒(1)): Assume (3). Let $\pi \in \mathbb{R}^d$ be an $(\alpha, \mathcal{R})$-good deal. Then the convex set $C = \mathrm{conv}(\bigcup_{\mathbb{P}\in\mathcal{R}} H \cap \mathrm{supp}\mu_{\mathbb{P}}) \subseteq H$, where $H = \{y \in \mathbb{R}^d \mid y \cdot \pi \geq 0\}$, satisfies the conditions in

[5] For any Borel set $A \subseteq \mathbb{R}^d$ with $\mu(A) \geq \alpha$, we have $\mu(A \cap \mathrm{supp}(\mu)) \geq \alpha$.

(3). Hence, there exists a martingale measure $\mathbb{Q}$ such that $\mathrm{supp}\mu_{\mathbb{Q}} \subseteq \overline{C}$ and $\mathbb{Q}(S_1 - S_0 \in \mathrm{ri}C) > 0$. Therefore, we have

$$0 = \pi \cdot \mathbb{E}_{\mathbb{Q}}[S_1 - S_0 \mid \mathcal{F}_0], \quad \mathbb{Q}\text{-a.s.} \tag{9}$$

but since $\mathbb{Q}(S_1 - S_0 \in \mathrm{ri}C) > 0$ we have

$$\mathbb{E}_{\mathbb{Q}}[\pi \cdot (S_1 - S_0) \mid \mathcal{F}_0] > 0, \quad \text{with positive } \mathbb{Q}\text{-probability}, \tag{10}$$

where for (9) we have used that $\mathbb{Q}$ is a martingale measure and for (10) we have used that $\overline{C} \subseteq H$ and $\mathrm{ri}C \subseteq \mathrm{ri}H = \{y \in \mathbb{R}^d \mid \pi \cdot y > 0\}$. Clearly, (10) is a contradiction to (9).

Step 4 ((2)⇒(3)): Assume (2). Let $C \subseteq \mathrm{conv}(\bigcup_{\mathbb{P}\in\mathcal{R}} \mathrm{supp}\mu_{\mathbb{P}})$ be convex with $\mathbb{P}(S_1 - S_0 \in \bar{C}) \geq \alpha$ for all $\mathbb{P} \in \mathcal{R}$. By (2), $0 \in \mathrm{ri}C$ and, without loss of generality, we can assume that there exists $\omega \in \Omega$ such that $S_1(\omega) - S_0(\omega) = 0 \in \mathrm{ri}C$. Let

$$\mathcal{C} = \Big\{\mathbb{Q} \in \mathcal{M}_1(\Omega) \mid \mathrm{supp}\mu_{\mathbb{Q}} \subseteq \bar{C}, \mathbb{Q}(S_1 - S_0 \in \mathrm{ri}C) > 0, \int_\Omega S_1^i d\mathbb{Q} < \infty, i = 0, \dots, d\Big\}. \tag{11}$$

Claim 1: $\mathcal{C} \neq \emptyset$.

Proof of Claim 1. The point measure δ_ω for some $\omega \in (S_1 - S_0)^{-1}(\mathrm{ri}C) \neq \emptyset$ is in $\mathcal{C}$, since the S_1^i, $i = 0, \dots d$ are finitely valued. □

Obviously, the set $\mathcal{C}$ is convex and we can now proceed by using the typical separation argument to show that $\mathcal{C}$ must contain a martingale measure for the market $(\Omega, \mathcal{F}, S_0, S_1)$. Therefore, let $R_i(y) = y_i - S_0^i$, $i = 1, \dots, d$. Define the nonempty convex subset of $\mathbb{R}^d$ as

$$K := \Big\{\Big(\int R_1 d\mu_{\mathbb{P}}, \dots, \int R_d d\mu_{\mathbb{P}}\Big) : \mathbb{Q} \in \mathcal{C}\Big\}. \tag{12}$$

There exists a martingale measure in $\mathcal{C}$, if and only if $0 \in K$. To achieve a contradiction, assume $0 \notin K$. By the separation theorem in $\mathbb{R}^d$, there exists a vector $0 \neq \phi \in \mathbb{R}^d$ such that $\phi \cdot k \geq 0$ for all

$k \in K$ and $\phi \cdot k_0 > 0$ for some $k_0 \in K$. Hence,

$$\int_{\mathbb{R}^d} \phi \cdot R d\mu_{\mathbb{Q}} \geq 0, \tag{13}$$

for all $\mathbb{Q} \in \mathcal{C}$, and

$$\int_{\mathbb{R}^d} \phi \cdot R d\mu_{\mathbb{Q}} > 0, \tag{14}$$

for some $\mathbb{Q} \in \mathcal{C}$.

Claim 2: $\forall \mathbb{P} \in \mathcal{R}, \mathbb{P}(S_1 - S_0 \in \bar{H}_\phi) \geq \alpha$, *where* $\bar{H}_\phi := \{y \in \mathbb{R}^d \mid \pi \cdot y \geq 0\}$.

Proof of Claim 2. To achieve a contradiction, assume that there exists $\mathbb{P} \in \mathcal{R}$ such that $\mathbb{P}(S_1 - S_0 \in \bar{H}_\phi) < \alpha$ or equivalently $\mathbb{P}(S_1 - S_0 \in \bar{H}_\phi^c) > 1 - \alpha$. Since for all $\mathbb{P} \in \mathcal{R}$ we have $\mathbb{P}(S_1 - S_0 \in \bar{C}) \geq \alpha$, we must have $C \cap \bar{H}_\phi^c \neq \emptyset$, because if $\bar{C} \cap \bar{H}_\phi^c = \emptyset$, then we would have $\mathbb{P}(S_1 - S_0 \in \bar{H}_\phi^c \cup \bar{C}) = \mathbb{P}(S_1 - S_0 \in \bar{H}_\phi^c) + \mathbb{P}(S_1 - S_0 \in \bar{C}) > 1$, which is a contradiction. That is, there exists $y \in \bar{C}$ with $\phi \cdot y < 0$ and since $(S_1 - S_0)(\Omega)$ is dense in $\bar{C}$, there must exists $\omega^* \in \Omega$ such that $\phi \cdot (S_1(\omega^*) - S_0(\omega^*)) < 0$. Let $\omega \in (S_1 - S_0)^{-1}(\mathrm{ri} C)$. Choose $p \in (0,1)$ such that $(1-p)\phi \cdot (S_1(\omega) - S_0) < p \left| \phi \cdot (S_1(\omega^*) - S_0) \right|$. Then the probability measure $\mathbb{Q} := p\delta_{\omega^*} + (1-p)\delta_\omega$ is in $\mathcal{C}$ and satisfies $\int_\Omega \phi \cdot (S_1 - S_0) d\mathbb{Q} < 0$, which is a contradiction to (13). $\square$

Claim 3: $\exists \mathbb{P} \in \mathcal{R}$, $\mathbb{P}(\phi \cdot (S_1 - S_0) > 0) > 0$.

Proof of Claim 3. For contradiction, assume $\mathbb{P}(\phi \cdot (S_1 - S_0) > 0) = 0$ for all $\mathbb{P} \in \mathcal{R}$. Choose $\mathbb{Q} \in \mathcal{C}$ satisfying (14), i.e. such that $\mathbb{Q}(\phi \cdot (S_1 - S_0)) > 0$. Then there exists $y \in \mathrm{supp}(\mu_{\mathbb{Q}})$ such that $\phi \cdot y > 0$. Since $\mathrm{supp}\mu_{\mathbb{Q}} \subseteq \bar{C} \subseteq \overline{\mathrm{conv}}(\bigcup_{\mathbb{P} \in \mathcal{R}} \mathrm{supp}\mu_{\mathbb{P}})$, we have two cases: First case is $y \in C$, that is $y = \alpha_1 y_1 + \cdots + \alpha_n y_n$ for some convex combination with $y_1, \ldots, y_n \in \bigcup_{\mathbb{P} \in \mathcal{R}} \mathrm{supp}(\mu_{\mathbb{P}})$. Then,

$$0 < \phi \cdot y = \alpha_1 \phi \cdot y_1 + \cdots + \alpha_n \phi \cdot y_n \tag{15}$$

and therefore $\phi \cdot y_i > 0$ for some $i \in \{1, \ldots, n\}$, which contradicts the assumption that $\phi \cdot y = 0$ $\mu_{\mathbb{P}}$-a.s. for all $\mathbb{P} \in \mathcal{R}$. Therefore, there must exist $\mathbb{P} \in \mathcal{R}$ such that $\mathbb{P}(\phi \cdot (S_1 - S_0) > 0) > 0$.

The second case is $y \in \bar{C} \setminus C$. But since the map $T : \mathbb{R}^d \to \mathbb{R}$ defined by $T(x) = \phi \cdot x$ for $x \in \mathbb{R}$ is continuous and the set C is convex, hence connected, there exists $y^* \in C$ such that $T(y^*) = \phi \cdot y^* > 0$ and this reduces the proof to the first case. $\square$

Claim 2 and Claim 3 show that $\phi \in \mathbb{R}^d$ is an $(\alpha, \mathcal{R})$-good deal, which contradicts our assumption that $(\Omega, \mathcal{F}, S_0, S_1)$ is free of $(\alpha, \mathcal{R})$-good deals. Consequently, $0 \in K$. This proves (3)$\Rightarrow$(2), which concludes the proof of Theorem 1. $\square$

So far, we have stated the FTAP under the absence of $(\alpha, \mathcal{R})$-good deals. In a next step, we will fix $\alpha = 1$ and reformulate the FTAP under absence of $\mathcal{R}$-arbitrage. To do so, we first need the following lemma.

Lemma 1: *Let $(\Omega, \mathcal{F})$ be a measurable space and $X : \Omega \to \mathbb{R}^d$ a measurable map such that $X_i(\omega) \geq -a$ for all $\omega \in \Omega$, all $i = 1, \ldots, d$ for some $a \in [0, \infty)$. Let $A \subseteq \overline{X(\Omega)}$ be a nonempty, closed subset. Assume that there exists $\omega_0 \in \Omega$ with $X(\omega_0) \in \mathrm{ri}(\mathrm{conv}(A))$. Then, there exists a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$, such that*

1. $A \cup \{X(\omega_0)\} = \mathrm{supp}(\mu_{\mathbb{P}}) \subseteq \overline{X(\Omega)}$
2. $\int_\Omega X_i d\mathbb{P} < \infty$, $i = 0, \ldots, d$,
3. $\mathbb{P}(X \in \mathrm{ri}(\mathrm{conv}(A))) > 0$.

Proof. Let $\mathcal{B}$ be a countable basis for A. Each $B \in \mathcal{B}$ contains an element $y_B \in X(\Omega)$. For $B \in \mathcal{B}$, let $\omega_B = X^{-1}(y_B)$. Let $(a_B)_{B \in \mathcal{B}} \subseteq (0,1)$ such that $\sum_{B \in \mathcal{B}} a_B + b = 1$, where $b \in (0,1)$. Then the probability measure $\mathbb{Q} = b\delta_{\omega_0} + \sum_{B \in \mathcal{B}} a_B \delta_{\omega_B}$, where δ_ω denotes the dirac measure at ω, satisfies (1) and (3). Define the constant $c := (\int_\Omega \frac{1}{1+a+M(\omega)} d\mathbb{Q}(\omega))^{-1}$, where $M(\omega) = \max_{i=0,\ldots,d} X_i(\omega)$. For $F \in \mathcal{F}$, define a probability measure by

$$\mathbb{P}(F) := \int_F \frac{c}{1 + a + M} d\mathbb{Q}.$$

Since the integrand is strictly positive, $\mathbb{P}$ also satisfies (1) and (3). Obviously, $\mathbb{P}$ also satisfies (2), hence $\mathbb{P}$ is the probability measure that satisfies the properties in Lemma 1. $\square$

Definition 5 ($\mathcal{R}$-full c-support measure): *A probability measure* $\mathbb{Q}$ *on the market model* $(\Omega, \mathcal{F}, S_0, S_1)$ *is said to have* $\mathcal{R}$-full c-support*, if*

$$\overline{\text{conv}}(\text{supp}(\mu_{\mathbb{Q}})) = \overline{\text{conv}}\Big(\bigcup_{\mathbb{P}\in\mathcal{R}} \text{supp}(\mu_{\mathbb{P}})\Big),^{6} \tag{16}$$

where $\mu_{\mathbb{Q}} = \mathbb{Q} \circ (S_1 - S_0)^{-1}$ *and* $\mu_{\mathbb{P}} = \mathbb{P} \circ (S_1 - S_0)^{-1}$ *for* $\mathbb{P} \in \mathcal{R}$*.*

In standard mathematical finance, i.e., when we are given only one prior $\mathcal{R} = \{\mathbb{P}\}$, any equivalent measure is also a $\mathcal{R}$-full c-support measure but not vice versa. From Definition 5, we see that a $\mathcal{R}$-full c-support measure depends on the price process through $(S_1 - S_0)$. In contrast, the property of an equivalent measure is independent of the price processes, since there exist measures which are non-equivalent but their support coincides.

In the next corollary, we state the FTAP under absence of $\mathcal{R}$-arbitrage, i.e., when $\alpha = 1$.

Corollary 1: *Let* $C^* = \text{conv}(\overline{\bigcup_{\mathbb{P}\in\mathcal{R}} \text{supp}\mu_{\mathbb{P}}})$ *and assume that* $S_1^i(\omega) - S_0^i(\omega) \geq -a$ *for some* $a \in [0, \infty)$*, all* $i = 1, \dots, d$ *and all* $\omega \in \Omega$*. The following are equivalent:*

1. $(\Omega, \mathcal{F}, S_0, S_1)$ *is free of* $\mathcal{R}$*-arbitrage.*

2. $0 \in \text{ri}C^*$.

3. *There exists a* $\mathcal{R}$*-full c-support martingale measure* $\mathbb{Q}$ *such that* $\mathbb{Q}(S_1 - S_0 \in \text{ri}C^*) > 0$.

Proof of Corollary 1. Equivalence of (1) and (2) and the implication (3)⇒(1) follow from Theorem 1. It remains only to show the implication (2)⇒(3). Assume (2), i.e., $0 \in \text{ri}C^*$. Without loss of generality, we can assume that there exists $\omega \in \Omega$ such that $S_1(\omega) - S_0(\omega) = 0 \in \text{ri}C^*$. Let $D := \overline{\bigcup_{\mathbb{P}\in\mathcal{R}} \text{supp}\mu_{\mathbb{P}}}$ and

$$\mathcal{C} = \Big\{\mathbb{Q} \in \mathcal{M}_1(\Omega) \mid \overline{\text{conv}}(\text{supp}\mu_{\mathbb{Q}}) = \overline{C^*}, \mathbb{Q}(S_1 - S_0 \in \text{ri}C^*) > 0, \int_{\Omega} S_1^i d\mathbb{Q} < \infty, i = 0, \dots, d\Big\}. \tag{17}$$

Claim 4: $\mathcal{C} \neq \emptyset$.

[6] Recall that for a subset $B \subseteq \mathbb{R}^d$, we have $\text{conv}(\overline{B}) \subseteq \overline{\text{conv}(B)} = \overline{\text{conv}}(B)$.

Proof of Claim 4. This is Lemma 1 for $X = S_1 - S_0$ and $A = D$. $\square$

Claim 5: $\forall \mathbb{P} \in \mathcal{R}, \mathbb{P}(S_1 - S_0 \in \bar{H}_\phi) = 1$, *where* $\bar{H}_\phi := \{y \in \mathbb{R}^d \mid \pi \cdot y \geq 0\}$.

Proof of Claim 5. To achieve a contradiction, assume that there exists $\mathbb{P} \in \mathcal{R}$ such that $\mathbb{P}(S_1 - S_0 \in \bar{H}_\phi) < 1$ or equivalently $\mathbb{P}(S_1 - S_0 \in \bar{H}_\phi^c) > 0$. Since for all $\mathbb{P} \in \mathcal{R}$, we have $\mathbb{P}(S_1 - S_0 \in \overline{C^*}) = 1$, we must have $\overline{C^*} \cap \bar{H}_\phi^c \neq \emptyset$, otherwise if $\overline{C^*} \cap \bar{H}_\phi^c = \emptyset$, then we would have $\mathbb{P}(S_1 - S_0 \in \bar{H}_\phi^c \cup \overline{C^*}) = \mathbb{P}(S_1 - S_0 \in \bar{H}_\phi^c) + \mathbb{P}(S_1 - S_0 \in \overline{C^*}) > 1$, which is a contradiction. Since $\mathrm{conv}((S_1 - S_0)(\Omega)) \cap \bar{C}^*$ is dense in $\overline{C}^*$ and $\bar{H}_\phi^c$ is open, there must exist $\omega^* \in \Omega$ such that $S_1(\omega^*) - S_0(\omega^*) \in \bar{C}^* \cap \bar{H}_\phi^c$, hence $\phi \cdot (S_1(\omega^*) - S_0(\omega^*)) < 0$. Further, let $\omega \in (S_1 - S_0)^{-1}(\{0\})$, such that $0 = S_1(\omega) - S_0(\omega) \in \mathrm{ri} C^*$. Choose a sequence $(\omega_k)_{k \in \mathbb{N}} \subset \Omega$ as in Lemma 1. Without loss of generality, we can assume $\omega_1 = \omega^*$ and $\omega_2 = \omega$, since otherwise we can just include this two elements into the sequence. Choose a sequence $(a_k)_{k \in \mathbb{N}} \subseteq (0, \infty)$, such that $\sum_{k=1}^\infty a_k = 1$ and

$$\Big| \frac{\phi \cdot R(\omega_1) a_1}{1 + M(\omega_1)} \Big| > c_0 \sum_{k=2}^\infty a_k, \tag{18}$$

where $c_0 = \sum_{i=1}^d |\phi_i|$. Define a probability measure by

$$\mathbb{P}_0^* = \sum_{k=1}^\infty a_k \delta_{\omega_k}. \tag{19}$$

Further define for $F \in \mathcal{F}$

$$\mathbb{P}_1^*(F) = \int_F \frac{c}{1 + a + M} d\mathbb{P}_0^*, \tag{20}$$

where $c \in \mathbb{R}$ is the normalizing constant. Clearly, $\mathbb{P}_1^* \in \mathcal{C}$. Then,

$$\begin{aligned} \int_\Omega \phi \cdot R(\omega) d\mathbb{P}_1^* &= \sum_{k=1}^\infty \phi \cdot R(\omega_k) \frac{1}{1 + M(\omega_k)} a_k & (21)\\ &= \frac{\phi \cdot R(\omega_1) a_1}{1 + M(\omega_1)} + \sum_{k=2}^\infty \frac{\sum_{i=1}^d \phi_i (S_i(\omega_k) - 1)}{1 + M(\omega_k)} a_k & (22)\\ &\leq \frac{\phi \cdot R(\omega_1) a_1}{1 + M(\omega_1)} + c_0 \sum_{k=2}^\infty a_k < 0 & (23) \end{aligned}$$

where in (23) we have used (18) and the assumption that $\phi \cdot R(\omega_1) < 0$. Now (23) is a contradiction to (13) and this proves Claim 5. $\square$

CLAIM 6: $\exists \mathbb{P} \in \mathcal{R}$, $\mathbb{P}(\phi \cdot (S_1 - S_0) > 0) > 0$.

Proof of Claim 6. Follows directly from Claim 3 in Theorem 1. $\square$

Claim 5 and Claim 6 show that $\phi \in \mathbb{R}^d$ is an $\mathcal{R}$-arbitrage, which contradicts our assumption that $(\Omega, \mathcal{F}, S_0, S_1)$ is $(\alpha, \mathcal{R})$-arbitrage free. Consequently, $0 \in K$ and this proves the implication (2)⇒(3). This proves the corollary. $\square$

2.2 Multi-period market

The one-period market model is now generalizable to a multi-period market model as in Föllmer and Schied (2011), Chapter 5. We consider a market model with d risky assets priced at times $t = 0, \ldots, T$. Let $(\Omega, \mathcal{F})$ be a measurable space and $\mathcal{F}_t$ a σ-subalgebra of $\mathcal{F}$ such that $\mathcal{F}_t \subseteq \mathcal{F}_{t'}$ for $t \leq t'$, $t, t' \in \{1, \ldots, T\}$. Let $S_t^i : \Omega \to \mathbb{R}$ be $\mathcal{F}_t$-measurable maps, $i = 1, \ldots d$ and $t = 0, \ldots, T$, representing the i-th risky asset at time t. Let $\mathcal{P}$ be any subset of the set $\mathcal{M}_1(\Omega)$ of all probability measures on $(\Omega, \mathcal{F})$.

To formulate the FTAP in a multi-period setting, we need to introduce the definition of a self-financing trading strategy, for which we follow the standard literature.

DEFINITION 6: *For $i = 1, \ldots, d$ and $t = 1, \ldots, T$ let $\xi_t^i : \Omega \to \mathbb{R}$ be $\mathcal{F}_{t-1}$-measurable maps. We call $\xi = (\xi_t)_{t=1,\ldots,T} = (\xi_t^1, \ldots, \xi_t^d)_{t=1,\ldots,T}$ a* trading strategy. *A trading strategy ξ is called* self-financing *if*

$$\xi_t \cdot S_t = \xi_{t+1} \cdot S_t \ \mathbb{P}\text{-}a.s. \text{ for all } \mathbb{P} \in \mathcal{P} \text{ and } t = 1, \ldots, T-1. \tag{24}$$

The value process *of a trading strategy ξ is*

$$V_0 := \xi_1 \cdot S_0 \text{ and } V_t := \xi_t \cdot S_t, \text{ for } t = 1, \ldots, T. \tag{25}$$

Similarly to the one-period case, we introduce $(\alpha, \mathcal{R})$-good deals in our multi-period market as follows.

DEFINITION 7: *Let $\alpha \in (0,1]$ and $\mathcal{R} \subseteq \mathcal{P}$ any subset. A self-financing trading strategy is called an* $(\alpha, \mathcal{R})$-good deal *if its value process V satisfies*

$$V_0 \leq 0, \quad \mathbb{P}(V_T \geq 0) \geq \alpha \textit{ for all } \mathbb{P} \in \mathcal{P} \textit{ and } \mathbb{P}(V_T > 0) > 0 \textit{ for some } \mathbb{P} \in \mathcal{P}. \tag{26}$$

DEFINITION 8: *A probability measure $\mathbb{Q}$ is called martingale measure for the market model $(\Omega, (\mathcal{F}_t)_{t=0,\ldots,T}, (S_t)_{t=0,\ldots,T})$ if for all $t \in \{1, \ldots, T\}$, we have*

$$E_{\mathbb{Q}}[S_t - S_{t-1} \mid \mathcal{F}_{t-1}] = 0 \ \mathbb{Q}\text{-}a.s.. \tag{27}$$

DEFINITION 9: *A probability measure $\mathbb{Q}$ for the market model $(\Omega, (\mathcal{F}_t)_{t=0,\ldots,T}, (S_t)_{t=0,\ldots,T})$ is said to have* $\mathcal{R}$-full c-support, *if for every $t \in \{1, \ldots, T\}$ it satisfies*

$$\overline{\text{conv}}(\text{supp}(\mu_{t,\mathbb{Q}})) = \overline{\text{conv}}\Big(\overline{\bigcup_{\mathbb{P} \in \mathcal{P}} \text{supp}(\mu_{t,\mathbb{P}})}\Big), \tag{28}$$

where $\mu_{t,\mathbb{Q}} = \mathbb{Q} \circ (S_t - S_{t-1})^{-1}$ and $\mu_{t,\mathbb{P}} = \mathbb{P} \circ (S_t - S_{t-1})^{-1}$ for $\mathbb{P} \in \mathcal{R}$.

In the one prior case, i.e. when $\mathcal{P}$ only consist one probability measure $\mathbb{P}$, absence of $\{\mathbb{P}\}$-arbitrage implies the existence of an equivalent martingale measure $\mathbb{Q}$, which then satisfies

$$\text{supp}(\mu_{t,\mathbb{Q}}) = \overline{\bigcup_{\mathbb{P} \in \mathcal{P}} \text{supp}(\mu_{t,\mathbb{P}})} = \text{supp}(\mu_{t,\mathbb{P}}), \tag{29}$$

for any $t \in \{1, \ldots, T-1\}$. Note that equation (29) holds for the same probability measure $\mathbb{Q}$ at every time instant t. Clearly, as in the one-period case, (29) does not imply that $\mathbb{Q}$ is equivalent to $\mathbb{P}$.

PROPOSITION 1: *There is an $(\alpha, \mathcal{R})$-good deal in the multi-period market model if and only if* *there exists* $t \in \{1, \dots, T\}$ *and a* $\mathcal{F}_{t-1}$*-measurable map* $\eta : \Omega \to \mathbb{R}^d$ *such that*

$$\forall \mathbb{P} \in \mathcal{P} : \; \mathbb{P}(\eta \cdot (S_t - S_{t-1}) \geq 0) \geq \alpha, \tag{30}$$

$$\exists \mathbb{P} \in \mathcal{P} : \mathbb{P}(\eta \cdot (S_t - S_{t-1}) > 0) > 0. \tag{31}$$

Proof. The proof is similar to Föllmer and Schied (2011), Prop. 5.11. Let ξ be an $(\alpha, \mathcal{R})$-good deal and V its value process. Define

$$t := \min\{k \mid \mathbb{P}(V_k \geq 0) \geq \alpha \;\text{ for all } \mathbb{P} \in \mathcal{R}, \text{ and } \mathbb{P}(V_k > 0) > 0 \text{ for some } \mathbb{P} \in \mathcal{R}\}. \tag{32}$$

Then, by assumption $t \leq T$ and we have the following two cases

1. either $\forall \mathbb{P} \in \mathcal{R} : \mathbb{P}(V_{t-1} \geq 0) \geq \alpha$ and $\mathbb{P}(V_{t-1} > 0) = 0$
2. or $\exists \mathbb{P} \in \mathcal{R}, \; \mathbb{P}(V_{t-1} \geq 0) < \alpha$,

since t was chosen minimal. Hence, in the first case, it follows that $\mathbb{P}(V_{t-1} = 0) \geq \alpha$ for all $\mathbb{P} \in \mathcal{R}$ and therefore

$$\forall \mathbb{P} \in \mathcal{R} : \mathbb{P}(\xi_t \cdot (S_t - S_{t-1}) = V_t - V_{t-1} = V_t) = \mathbb{P}(V_{t-1} = 0) \geq \alpha. \tag{33}$$

Thus, $\eta := \xi_t$ satisfies (30) and (31). In the second case, we take $\eta := \xi_t 1_{\{V_{t-1} < 0\}}$, which is $\mathcal{F}_{t-1}$-measurable and satisfies

$$\eta(\omega)(S_t(\omega) - S_{t-1}(\omega)(\omega) = (V_t(\omega) - V_{t-1}(\omega))1_{\{V_{t-1}<0\}}(\omega) \geq -V_{t-1}(\omega)1_{\{V_{t-1}<0\}}(\omega) \geq 0 \tag{34}$$

for every $\omega \in \Omega$. By 2., there exists $\mathbb{P} \in \mathcal{R}$ such that $\mathbb{P}(V_{t-1} < 0) > 1 - \alpha > 0$, hence the random variable on the right hand side of (34) is strictly positive with positive probability for some $\mathbb{P} \in \mathcal{R}$. This proves necessity.

Now we prove sufficiency: For t and η as in the statement of the proposition and satisfying (30) and (31), define

$$\xi_s = \begin{cases} \eta & \text{, if } s = t \\ 0 & \text{, else.} \end{cases}$$

Then ξ is a trading strategy which is also an $(\alpha, \mathcal{R})$-good deal. $\square$

Theorem 2 (Multi-period FTAP under multiple priors): *The following are equivalent:*

1. *The market model $(\Omega, (\mathcal{F}_t)_{t=0,\dots,T}, (S_t)_{t=0,\dots,T})$ is free of $(\alpha, \mathcal{R})$-good deals.*

2. *For any $t \in \{1, \dots, T\}$ and any convex set $C \subseteq \mathbb{R}^d$ with $\mathbb{P}(S_t - S_{t-1} \in \bar{C}) \geq \alpha$ for all $\mathbb{P} \in \mathcal{R}$, we have $0 \in \mathrm{ri}C$.*

3. *For any $t \in \{1, \dots, T\}$ and any convex set $C \subseteq \mathrm{conv}(\bigcup_{\mathbb{P}\in\mathcal{R}} \mathrm{supp}\mu_{\mathbb{P},t})$ with $\mathbb{P}(S_t - S_{t-1} \in \bar{C}) \geq \alpha$ for all $\mathbb{P} \in \mathcal{R}$, there exists a martingale measure $\mathbb{Q}_t$ for the market model $(\Omega, (\mathcal{F}_{t'})_{t'=t-1,t}, S_{t-1}, S_t)$, such that $\mathrm{supp}\mu_{\mathbb{Q}_t} \subseteq \bar{C}$ and $\mathbb{Q}_t(S_t - S_{t-1} \in \mathrm{ri}C) > 0$.*

Proof. This follows from Theorem 1 and Proposition 1. $\square$

By Corollary 1, we know that if there is no $\mathcal{R}$-arbitrage in a one-period market model, then there exist a $\mathcal{R}$-full support martingale measure for this market model. But the existence of an $\mathcal{R}$-full support martingale measure between time $t-1$ and t for all $t \in \{1, \dots, T\}$ does not guarantee the existence of a martingale measure for the whole multi-period market model which has $\mathcal{R}$-full support at every time instant $t \in \{1, \dots, T\}$. Therefore, for the multi-period market we obtain the following corollary for the case $\alpha = 1$.

Corollary 2: *Assume that trading strategies and the risky assets are bounded. For $t \in \{1, \dots, T\}$, let $C_t^* = \mathrm{conv}(\overline{\bigcup_{\mathbb{P}\in\mathcal{R}} \mathrm{supp}\mu_{t,\mathbb{P}}})$, where $\mu_{t,\mathbb{P}} = \mathbb{P} \circ (S_t - S_{t-1})^{-1}$. The following are equivalent:*

1. *The market model $(\Omega, (\mathcal{F}_t)_{t=0,\dots,T}, (S_t)_{t=0,\dots,T})$ is free of $\mathcal{R}$-arbitrage.*

2. *For all $t \in \{1, \dots, T\}$ we have $0 \in \mathrm{ri}C_t^*$.*

3. *For every $t \in \{1, \dots, T\}$, there exist a $\mathcal{R}$-full c-support martingale measure $\mathbb{Q}_t$ for the market model $(\Omega, (\mathcal{F}_{t'})_{t'=t-1,t}, S_{t-1}, S_t)$ with $\mathbb{Q}_t(S_t - S_{t-1} \in \mathrm{ri}C_t^*) > 0$.*

Proof. The above result follows from Theorem 1 and Proposition 1. □

We note that the proof of the standard FTAP fails in the multiple-priors case due to lack of a reference measure, i.e., a fixed, single prior. The main difference between an equivalent martingale measure in a one-prior market model $(\Omega, \mathcal{F}, \mathbb{P})$ and a martingale measure $\mathbb{Q}$ with $\mathcal{P}$-full support in a multiple-priors market model $(\Omega, \mathcal{F}, \mathcal{P})$ is, up to the fact that they are both martingale measures, that equivalence is only a property of $\mathbb{P}$, while $\mathcal{P}$-full support is a property of $\mathcal{P}$ and the stochastic process $(S_t)_{t=0,\ldots,T}$. This difference causes the main difficulties in obtaining results analogous to the standard FTAP. Therefore, it remains an open question under which conditions on the set of priors $\mathcal{P}$ and the stochastic process $(S_t)_{t=0,\ldots,T}$ there exists a martingale measure for the whole multi-step market model which satisfies the $\mathcal{P}$-full support property at every instant t.

3 FTAP without Priors

In this section, we present the FTAP for the case when there is no prior at all. For comparison with previous results, e.g., Riedel (2011), we start as in the previous section with the one-period market and extend the result subsequently to a multi-period market.

3.1 One-period market

Let $(\Omega, \mathcal{F})$ be a measurable space and $S_0, S_1 : \Omega \to \mathbb{R}^d$ be measurable maps. Assume that $S_1^i(\omega) - S_0^i(\omega) \geq -a$ for some $a \in [0, \infty)$, all $i = 1, \ldots d$ and all $\omega \in \Omega$.

Definition 10: *A* portfolio *is a vector* $\pi \in \mathbb{R}^d$. *A portfolio* π *is called a* deterministic arbitrage, *if we have*

$$\pi \cdot (S_1(\omega) - S_0(\omega)) \geq 0, \text{ for all } \omega \in \Omega \tag{35}$$

$$\pi \cdot (S_1(\omega) - S_0(\omega)) > 0, \text{ for some } \omega \in \Omega. \tag{36}$$

We say that the market model $(\Omega, \mathcal{F}, S_0, S_1)$ *is free of deterministic arbitrage, if there does not exist a deterministic arbitrage portfolio* π.

Definition 11: *A probability measure* $\mathbb{Q}$ *on the market model* $(\Omega, \mathcal{F}, S_0, S_1)$ *is said to have* full support *if* $\operatorname{supp}\mu_{\mathbb{Q}} = \overline{(S_1 - S_0)(\Omega)}$.

Theorem 3 (One-period FTAP under no priors): *Let* $C = \operatorname{conv}(\overline{(S_1 - S_0)(\Omega)})$. *The following are equivalent:*

1. *There is no deterministic arbitrage.*

2. $0 \in \mathrm{ri}C$.

3. *There exists a full support martingale measure* $\mathbb{Q}$ *such that* $\mathbb{Q}(S_1 - S_0 \in \mathrm{ri}C) > 0$.

Proof. We first prove (1)⇒(2). Assume $0 \notin \mathrm{ri}C$, $C := \operatorname{conv}(\overline{(S_1 - S_0)(\Omega)})$. Then by the separation theorem, there is a deterministic arbitrage. To prove (2)⇒(1), we assume $0 \in \mathrm{ri}(\operatorname{conv}(\overline{(S_1 - S_0)(\Omega)}))$ and let $\pi \in \mathbb{R}^d$ be a deterministic arbitrage. Then, $\pi \cdot \phi \geq 0$ for all $\phi \in C$, and there is $\phi' \in C$ such that $\pi \cdot \phi > 0$. Then, $-\varepsilon\phi \in C$ for some $\varepsilon > 0$, hence $-\varepsilon\pi \cdot \phi < 0$ which is a contradiction to $\pi \cdot y \geq 0$ for all $y \in C$. Finally, (2)⇔(3) follows from Corollary 1 and Lemma 1. □

To see the advantages of Corollary 1 over the corresponding theorems given in Riedel (2011) and Cherny (2007), we restate their theorems here:

Theorem 4 (Riedel (2011)): *Assume that* Ω *is a Polish space,* $\mathcal{F}$ *is the Borel* σ*-algebra of* Ω *and* S_1 *is continuous with* $S_1^i \geq 0$ *for all* $i = 1, \ldots, d$ *and* $S_0 \in \mathbb{R}^d$ *is constant. Then there is no deterministic arbitrage, if and only if there exists a martingale measure assigning positive values to every open set in* Ω.

Theorem 5 (Cherny (2007)): *Let* $(\Omega, \mathcal{F})$ *be a measurable space. Let* $S_1 : \Omega \to \mathbb{R}^d$ *be measurable and* $S_0 \in \mathbb{R}^d$ *be constant. The following are equivalent:*

1. $(\Omega, \mathcal{F}, S_0, S_1)$ *is deterministic-arbitrage free,*

2. $0 \in \mathrm{ri}(C)$, *where* $C = \operatorname{conv}(\overline{(S_1 - S_0)(\Omega)})$,

3. *For all* $F \in \mathcal{F} \setminus \emptyset$ *there exists a martingale measure* $\mathbb{P}$ *such that* $\mathbb{P}(F) > 0$.

Here is an example of a compact Hausdorff space, on which there does not exist a probability measure assigning positive values to all open sets, i.e., there does not exist a full c-support measure as in Riedel (2011). It shows that the characterization of Riedel (2011) is not valid for general spaces:

EXAMPLE 1: Let Ω be an uncountable set equipped with the discrete topology. Denote by $X = \beta\Omega$ the Stone-Cech compactification of Ω. Then each singleton $\{\omega\}$, $\omega \in \Omega$, is open in X. Therefore, any measure that charges every open set must have total mass infinity, it cannot even be σ-finite. Hence, for such a space there does not exist a probability measure, which assigns positive values to all nonempty open sets.

Comparing property 3 of Corollary 1 with property 3 of Theorem 5, we see that in Cherny's theorem one has to check for every nonempty measurable set the existence of a martingale measure, which values this set positively. In contrast, our Corollary 1 only requires us to check, if there exists one martingale measure, which values the measurable set $F = (S_1 - S_0)^{-1}(\mathrm{ri}C)$ positively. Example 1 also shows the impossibility of a probability measure which values all nonempty measurable sets positively. Consequently, already the weaker condition 3 of Corollary 1 is necessary and sufficient for the absence deterministic arbitrage.

3.2 Multi-period market

In this section, let $(\Omega, \mathcal{F})$ be a measurable space and and $(\mathcal{F}_t)_{t=0,\dots,T}$ a sequence of sub-σ-algebras of $\mathcal{F}$ with $\mathcal{F}_{t'} \subseteq \mathcal{F}_t$ for all $t', t \in \{0, \dots, T\}$, $t' \leq t$. Further, for $t \in \{0, \dots, T\}$ let $S_t : \Omega \to \mathbb{R}^d$ be measurable maps.

DEFINITION 12: *We say that there is a* deterministic arbitrage *in the market model* $(\Omega, (\mathcal{F}_t)_{t=0,\dots,T}, (S_t)_{t=0,\dots,T})$, *if there exists a trading strategy* $(\xi_t)_{t=1,\dots,T}$ *such that its value process* V *satisfies*

$$V_0 \leq 0,\ V_T(\omega) \geq 0 \text{ for all } \omega \in \Omega \text{ and } V_T(\omega) > 0 \text{ for some } \omega \in \Omega. \tag{37}$$

We say that the market model $(\Omega, (\mathcal{F}_t)_{t=0,\dots,T}, (S_t)_{t=0,\dots,T})$ *is free of deterministic arbitrage, if there does not exist a deterministic arbitrage trading strategy.*

Remark 1: Similarly as in Proposition 1, one can show that (37) is equivalent to the existence of a $\mathcal{F}_{t-1}$-measurable random variable η such that $\eta \cdot (S_t(\omega) - S_{t-1}(\omega)) \geq 0$ for all $\omega \in \Omega$ and $\eta \cdot (S_t(\omega) - S_{t-1}(\omega)) > 0$ for some $\omega \in \Omega$, where $t \in \{1, \ldots, T\}$.

Definition 13: *A probability measure on the market model $(\Omega, (\mathcal{F}_t)_{t=0,\ldots,T}, (S_t)_{t=0,\ldots,T})$ is said to have* full c-support *if for all $t \in \{1, \ldots, T\}$:*

$$\text{supp}\mu_{\mathbb{Q},t} = \overline{(S_t - S_{t-1})(\Omega)} \tag{38}$$

where $\mu_{\mathbb{Q},t} = \mathbb{Q} \circ (S_t - S_{t-1})^{-1}$.

Lemma 2: *Suppose that for all $t \in \{1, \ldots, T\}$ we have $0 \in \text{ri}C_t$, where $C_t := \text{conv}(\overline{(S_t - S_{t-1})(\Omega)})$. Then there exists a martingale measure $\mathbb{Q}$ such that*

$$\mathbb{Q}(S_t - S_{t-1} \in \text{ri}C_t) > 0 \textit{ for all } t \in \{1, \ldots, T\}. \tag{39}$$

Proof. Assume that for all $t \in \{1, \ldots, T\}$ there exists $\omega_t \in \Omega$ such that $S_t(\omega_t) - S_{t-1}(\omega_t) = 0 \in \text{ri}C_t$. Let $\alpha_t \in (0, 1]$ such that $\sum_{t=0}^{T} \alpha_t = 1$. Then the probability measure $\mathbb{Q} = \sum_{t=1}^{T} \alpha_t \delta_{\omega_t}$ is a martingale measure which satisfies (39). □

Theorem 6 (Multi-period FTAP under no priors): *For $t \in \{1, \ldots, T\}$ let $C_t := \text{conv}(\overline{(S_t - S_{t-1})(\Omega)})$. The following are equivalent:*

1. *There is no deterministic arbitrage.*
2. *For all $t \in \{1, \ldots, T\}$, we have $0 \in \text{ri}C_t$.*
3. *For each one-period market model $(\Omega, (\mathcal{F}_{t'})_{t'=t-1,t}, (S_{t'})_{t'=t-1,t})$, $t \in \{1, \ldots, T\}$, there exists a full c-support martingale measure $\mathbb{Q}_t$ such that $\mathbb{Q}_t(S_t - S_{t-1} \in \text{ri}C_t) > 0$.*

Proof. This follows from Theorem 3 and Remark 1. □

Note that in the above theorem, we only prove the existence of a full c-support martingale measure for each time-step, but not the existence of a full c-support martingale measure that spans the whole time period from 0 until T. Cherny (2007) derives a different version of the FTAP for the multi-period market model under no priors, for which a martingale measure over the whole period exists. However, he had to make some additional assumptions, such as Cherny (2007), Assumption 3.5, and the assumption that at every time t, $t = 0 \leq, T$ there exist finitely many atoms σ-algebra $\mathcal{F}_t$ covering whole Ω. For our setup, it remains an open question under which conditions on the market model $(\Omega, (\mathcal{F}_t)_{t=0,\dots,T}, (S_t)_{t=0,\dots,T})$, there exists a full c-support martingale measure.

4 Conclusion

We derived a fundamental theorem of asset pricing under the assumption of multiple-priors or under absence of any prior probability. Contrary to what was obtained so far, we formulated the theorem for measurable spaces and measurable functions without further assumptions. Our results, and in particular Theorem 1, may serve as a basis to further develop mathematical finance towards quasi-sure analysis on general measurable spaces, where the underlying probability measures are allowed to be non-equivalent to one another. A promising avenue of future research is the extension of our analysis to a continuous-time setting.

References

Chen, Z., and L. Epstein, 2002, "Ambiguity, risk, and asset returns in continuous time," *Econometrica*, 70(4), 1403–1443.

Cherny, A., 2007, "General arbitrage pricing model. III. Possibility approach," in *Séminaire de Probabilités XL*. Springer, Berlin, vol. 1899 of *Lecture Notes in Math.*, pp. 463–481.

Delbaen, F., and W. Schachermayer, 1994, "A general version of the fundamental theorem of asset pricing," *Mathematische annalen*, 300(1), 463–520.

——— , 2006, *The mathematics of arbitrage*. Springer Berlin.

Epstein, L., and S. Ji, 2013, "Ambiguous volatility, possibility and utility in continuous time," *arXiv preprint arXiv:1103.1652*.

Epstein, L. G., and M. Schneider, 2003, "Recursive multiple-priors," *Journal of Economic Theory*, 113(1), 1–31.

Föllmer, H., and A. Schied, 2011, *Stochastic finance*. Walter de Gruyter & Co., Berlin, extended edn., An introduction in discrete time.

Hansen, L. P., and T. J. Sargent, 2001, "Robust Control and Model Uncertainty," *The American Economic Review*, 91(2), pp. 60–66.

Harrison, J., and S. R. Pliska, 1981, "Martingales and stochastic integrals in the theory of continuous trading," *Stochastic Processes and their Applications*, 11(3), 215 – 260.

Ju, N., and J. Miao, 2012, "Ambiguity, learning, and asset returns," *Econometrica*, 80(2), 559–591.

Knight, F., 1921, "Risk, uncertainty and profit," .

Leippold, M., F. Trojani, and P. Vanini, 2008, "Learning and asset prices under ambiguous information," *Review of Financial Studies*, 21(6), 2565–2597.

Riedel, F., 2011, "Finance Without Probabilistic Prior Assumptions," *ArXiv e-prints*.

Roubini, N., 2007, "Current market turmoil: Non-priceable Knightian "uncertainty" rather than priceable market "risk"," *EconoMonitor*, August, 15th.

Ulrich, M., 2013, "Inflation ambiguity and the term structure of U.S. Government bonds," *Journal of Monetary Economics*, 60(2), 295 – 309.